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SOME DYNAMICAL PROPERTIES OF SEQUENTIALLY ACQUIRED INFORMATION

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Abstract—In this article we obtain two closely related theorems that essentially say that, no matter what information metric is used, on the average the value of the accumulated information at stopping time is bounded by a multiple of the expected stopping time. These results are also independent of the particular stopping strategy employed, although they do require that the expected stopping time be finite. These results, along with a general type of stopping strategy based on incremental information, are given. Later we apply our general theorem to a specific stopping strategy associated with the GIS model. Although we concentrate on the problem of stopping, the information function on which this stopping decision is based can also be used to choose the COA for the next cycle of the feedback loop. We will apply our results to an estimation problem involving the well-known Shannon-Wiener measure of information. Since our theorems require that the expected stopping times be finite, some time is devoted to a discussion of necessary and sufficient conditions for finite expected stopping times.

1. INTRODUCTION

Consider a situation in which information is being fed back to a Decision Maker (DM) in a cyclical or sequential manner. Each cycle through the feedback loop conveys the same type of information, although the value of the information, as measured by some information function, may vary from cycle to cycle. At the conclusion of each cycle, the DM must decide whether to stop the process or continue through another cycle. The criteria he uses to make this decision is called his stopping strategy.

In some instances the stop-continue decision is the only function of the DM. Such a situation could occur if the DM were using the information to sequentially update the estimate of some unknown parameter. Stopping would normally occur when the DM becomes convinced that further information will not significantly increase the precision of his estimate. In other instances, the DM must choose among several competing hypotheses at the termination of a feedback loop. The general strategy in this situation is to accrue information until the correct hypothesis can be chosen with high probability. This particular version of the problem is generally referred to as sequential hypothesis testing and has received much attention in the statistical literature. Pioneering work was

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done in this area by Wald [8] who developed the widely-used Sequential Probability Ratio Test (SPRT). Other important contributions to the development of this area have been made by Robbins [5] and Sobel [7]. Readers interested in the general area of sequential hypothesis testing are referred to Ghosh [2].

Sometimes the DM must make intermediate decisions prior to each cycle of the feedback loop, as well as the major decisions of when to stop and what final course of action to take after stopping. An example of this situation is a sequential clinical trial in which each incoming patient must be assigned to one of several methods of treatment. This particular topic, which is really a specialized version of sequential hypothesis testing, has also generated numerous papers. For a review of this literature see [3].

The general situation in which the DM must initiate each cycle of the feedback loop by choosing among several competing courses of action (COA's) has been modeled by Yovits *et al.* [10–12]. In the generalized information system (GIS) proposed in [10], the DM makes his choices on the basis of "a priori" knowledge as well as on the information feedback to him on each cycle. Criteria for stopping the feedback cycle are not considered in [10]. However, Aló, Kleyle, and de Korvin [1] have developed several stopping strategies in conjunction with a specific selection mechanism for choosing among multiple COA's in the context of the GIS model.

In this article, we obtain two closely related theorems that essentially say that, no matter what information metric is used, on the average the value of the accumulated information at stopping time is bounded by a multiple of the expected stopping time. These results are also independent of the particular stopping strategy employed, although they do require that the expected stopping time be finite. These results, along with a general type of stopping strategy based on incremental information, are given. Later, we will apply our general theorem to a specific stopping strategy associated with the GIS model. Although we concentrate on the problem of stopping, the information function on which this stopping decision is based can also be used to choose the COA for the next cycle of the feedback loop. We will apply our results to an estimation problem involving the well-known Shannon-Wiener measure of information. Since our theorems require that the expected stopping time be finite, some time is devoted to a discussion of necessary and sufficient conditions for finite expected stopping times.

2. THEORY

Let $X_1, X_2, \ldots$ denote a stochastic process defined on a probability space (Ω, F, P). An extended stopping time is an extended integer-valued random variable T such that

$$(T = k) \in F(X_1, X_2, \ldots, X_k) \tag{1}$$

where $F(X_1, X_2, \ldots, X_k)$ denotes the σ-field generated by $(X_1, X_2, \ldots, X_k)$.

In our application of this concept to information theory, the elements of the stochastic process $X_1, X_2, \ldots$ will represent incremental information. That is, X_k will denote the value of the information obtained on the kth cycle through the feedback loop. T, of course, denotes the cycle or trial on which stopping occurs. Since no matter what the specific stopping strategy may be, the decision to stop on the kth cycle will always be a function of the information obtained on the first k cycles, so that stopping strategies will, at the very least, be extended stopping times.

If in addition to condition (1), $T < \infty$ a.s., T is called a stopping rule. Thus, a stopping strategy generates a bona fide stopping rule only if it will eventually terminate the feedback process with a probability of one.

The information accumulated through k cycles is

$$I_k = X_1 + X_2 + \cdots + X_k. \tag{2}$$

For the measure of information given by (2) to be truly cumulative, I_k must be non-decreasing, which requires that the incremental information $X_k \geqslant 0$ a.s. for all k. We will consider one such measure in the next section. However, if negative increments are allowed, virtually any information function can be written as above by simply defining

$$X_1 = I_1 \qquad X_k = I_k - I_{k-1} \qquad k \geqslant 2. \tag{3}$$

Since random variable I_k is an estimate of the true information function, a negative increment simply means that this estimate is being revised downward. Many well-known information functions depend on unknown parameters. Sequential estimation of these parameters based on the feedback data could cause such a downward revision. An example of this situation will be given in Sec. 3.

We now state the first of our two theorems on the expected value of the information accumulated by stopping time.

THEOREM 1. Let $X_1, X_2, \ldots$ be a stochastic process, and let T be a stopping time associated with this process. If I_T is given by (2) and if

(i) $E(T) < \infty$, and
(ii) $E[\,|X_k|\,|\,X_1, \ldots, X_{k-1}] \leqslant \gamma < \infty$ for all $k \leqslant T$, then

$$E(I_T) \leqslant \gamma\, E(T). \tag{4}$$

Proof.

$$\begin{aligned} E[I_T] &= \sum_{k=1}^{\infty} \int_{(T=k)} I_k \,\mathrm{d}P = \sum_{k=1}^{\infty} \sum_{j=1}^{k} \int_{(T=k)} X_j \,\mathrm{d}P \\ &= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \int_{(T=K)} X_j \,\mathrm{d}p = \sum_{j=1}^{\infty} \int_{(T\geqslant j)} X_j \,\mathrm{d}P. \end{aligned} \tag{5}$$

Now $(T \geqslant j) = (T < j)^c \in F(X_1, \ldots, X_{j-1})$ which along with condition (ii) implies that

$$\int_{(T\geqslant j)} X_j \,\mathrm{d}P \leqslant \int_{(T\geqslant j)} |X_j| \,\mathrm{d}P = \int_{(T\geqslant j)} E[\,|X_j|\,|\,X_1, \ldots, X_{j-1}] \leqslant \gamma P(T \geqslant k).$$

Thus,

$$E[I_T] \leqslant \gamma \sum_{j=1}^{\infty} P(T \geqslant j) = \gamma E(T). \tag{6}$$

The argument employed in the above proof is valid whether or not condition (i) holds. However, if condition (i) is not met, the conclusion is vacuous. The conditional expectation in condition (ii) may be difficult to compute in applications. However, if $|X_j| \leqslant \gamma$ a.s. for all $j \leqslant T$, then condition (ii) must hold. Condition (ii) is given in favor of the more easily verified condition because it is weaker and results in a slightly more general theorem.

The above theorem is applicable whether the increments are positive or negative, but it is most appropriate when $X_k \geq 0$ a.s. for all $k \leq T$. When negative increments are possible, the upper bound given by (4) may greatly exceed $E(I_T)$. The reason for this is that the right-hand side of (5) contains both positive and negative terms, whereas the series in (6) consists of positive terms only. If only non-negative increments are possible, I_k is monotonically increasing. This monotonicity is a desirable property for an information function to have, particularly if it is designed specifically for use in a sequential feedback system. However, many commonly used measures of information are not monotonic when applied to a sequential feedback system.

The following theorem is useful when the incremental information can be negative.

THEOREM 2. Let $X_1, X_2, \ldots$ be a stochastic process and let T be an associated stopping rule. Then for I_T given by (2), if

(i) $E(T) < \infty$,
(ii) There exist constants $\alpha \leq 0 \leq \beta$ such that $\alpha \leq X_k \leq \beta$ a.s. for all $k \leq T$;
(iii) $P(X_k \geq 0) = \pi^+$ $P(X_t < 0) = \pi^-$ for all $k \leq T$;
(iv) Events $(T \geq k)$ and $(X_k \geq 0)$ are independent for all $k \leq T$;

$$\text{then } \alpha\pi^- E(T) \leq E(I_T) \leq \beta\pi^- E(T) \tag{7}$$

Proof. From condition (ii)

$$0 \leq \int_A X_k \, dP \leq \beta P(T \geq k, X_k \geq 0) \quad \text{where} \quad A = (T \geq k) \cap (X_k \geq 0).$$

Using conditions (iii) and (iv) the above inequality can be rewritten:

$$0 \leq \int_A X_k \, dP \leq \beta\pi^- P(T \geq k). \tag{8}$$

In a similar manner we obtain

$$\alpha\pi^- P(T \geq k) < \int_B X_k \, dP \leq 0 \quad \text{where} \quad B = (T \geq k) \cap (X_k < 0). \tag{9}$$

Combining (8) and (9) yields

$$\alpha\pi^- P(T \geq k) \leq \int_{(T \geq k)} X_k \, dP \leq \beta\pi^- P(T \geq k) \tag{10}$$

Applying the inequality in (10) to the result in line (5) of Theorem 1, leads to the inequality in line (7).

When applied to the information feedback loop described in Sec. 1, both theorems give bounds on the expected value of the accumulated information in terms of the expected stopping times. Condition (ii) in Theorem 2 cannot be written in terms of conditional expectations as was its counterpart in Theorem 1 since

$$(T \geq k) \cap (X_k \geq 0) \in F(X_1, \ldots, X_k)$$

rather than in $F(X_1, \ldots, X_{k-1})$.

The above theorems are valid no matter what stopping strategy is used so long as it generates a stopping rule with finite expectation. However, in this article we will concentrate on stopping times defined in terms of the incremental information given by (3). Specifically, our stopping strategy is defined as follows:

Stopping strategy: Given a threshold $\delta > 0$, stop on the first trial T for which $|X_T| < \delta$.

Stopping strategies of the type defined above are obviously compatible with measures of information designed specifically for the type of feedback loop considered in this study. However, they sometimes work quite well when other information measures are adapted to this feedback system. An adaptation of the Shannon-Wiener information function is given in a later section.

3. APPLICATION TO THE YOVITS GIS MODEL

In the generalized information system proposed by Yovits *et al.* [10], the DM must choose from among m COA's, each of which may generate one of n possible outcomes. At the outset the DM does not know the probabilities with which the various COA's generate each of the n possible outcomes. Thus, the major unknown parameters are the conditional probabilities.

$$P_{ij} = P(\text{outcome} = o_j \mid \text{COA} = a_i)\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Initial estimtes of these conditional probabilities are provided by the DM, and his initial choice of the COA which commences the feedback process is based on this choice. After each cycle, these estimates are updated by incorporating the most recent information. Kleyle and de Korvin [4] propose an updated estimate of P_{ij} which after k complete cycles is given by

$$P_{kij} = \frac{\theta_{ij} + N_{kij}}{\theta_{i.} + N_{ki.}}. \tag{11}$$

θ_{ij} is the initial weight assigned to the action-outcome pair (a_i, o_j) by the DM. N_{kij} is a random variable which counts the number of times COA a_i results in outcome o_j through the first k trials.

$$\theta_{i.} = \sum_{j=1}^{n} \theta_{ij}, \qquad N_{ki.} = \sum_{j=1}^{n} N_{kij}.$$

Note that $N_{ki.}$ denotes the total number of times COA a_i is chosen in the first k cycles. Updated estimates of the type given by (11) were investigated in some detail in [1].

Since the information obtained on each trial is used to update the estimated conditional probabilities, the value of this information can be measured by the degree to which these estimates are changed. Thus, a reasonable measure of the incremental information associated with the kth trial is,

$$X_k = \max_i \max_j |\hat{P}_{kij} - \hat{P}_{k-1ij}| \qquad k \geqslant 1. \tag{12}$$

Obviously since $N_{ki.}$ changes only for the COA taken on the kth trial, the max over index

i will coincide with the index of the COA taken on trial k. Furthermore, it can be shown that the max over j occurs for the index matching that of the outcome which occurs on trial k.

The measure of the total information obtained on the first k trials is given by (2). From (12) it is clear that the increments are always positive, so that I_k is strictly increasing. This method of measuring information in a generalized feedback system was first proposed by Kleyle and de Korvin in [4]. Yovits *et al.* [10] propose a different measure of information which is used in connection with a randomized scheme for choosing among the different COA's.

Since the information function is monotonic we apply Theorem 1 in conjunction with the stopping rule given in Sec. 2. Condition (i) is verified by constructing an upper bound for $E(T)$. In [4] it is shown that X_k defined by (12) can be written as

$$X_k = \frac{1 - \hat{P}_{ki_0j_0}}{\theta_{i_0.} + N_{ki_0.} - 1}$$

where (a_{i_0}, O_{j_0}) denote the action-outcome pair associated with the kth cycle. Therefore,

$$X_k < \delta \Rightarrow N_{ki_0.} > 1 - \theta_{i_0.} + \frac{1 - \hat{P}_{i_0j_0}}{\delta}.$$

Now let M denote the *smallest* integer which equals or exceeds $1 - \min \theta_{i.} + \frac{1}{\delta}$. We then note that

$$N_{ki.} \geqslant M \Rightarrow N_{ki.} \geqslant 1 - \theta_{i.} + \frac{1}{\delta} > 1 - \theta_{i.} + \frac{1 - \hat{P}_{kij}}{\delta} \quad \text{for all} \quad i, j$$

Thus, stopping will occur on or before the trial on which one of the $N_{ki.}$'s equals M for the first time. The very latest this could possibly occur is on trial $m(M - 1) + 1$. It follows that

$$P[T \leqslant m(M - 1) + 1] = 1 \tag{13}$$

and condition (i) is satisfied.

Condition (ii) is trivial in this application since

$$|\hat{P}_{kij} - \hat{P}_{k-1ij}| < 1 \qquad \text{a.s. for all } i, j, k \text{ implies that}$$

$$X_k = \max_i \max_j |P_{kij} - P_{k-1ij}| < 1 \qquad \text{a.s. for all } k \leqslant T.$$

Thus $\gamma = 1$ and from line (4), Theorem 1,

$$E(I_T) \leqslant E(T). \tag{14}$$

Since I_k is strictly increasing, the bound given by (14) should be fairly sharp. Also, the bound makes intuitive sense in this example since the fact that each increment is less than 1 implies that $I_T < T$ a.s. The problem with getting an explicit expression for the right-hand side of (14) is that $E(T)$ is extremely difficult to evaluate. From (13) it is obvious that

$$E(T) < m(M - 1) + 1.$$

However, this bound, obtained by considering the longest possible number of cycles, will tend to greatly exceed $E(T)$.

Application to Shannon-Wiener information

The Shannon-Wiener measure of the information obtained from the occurrence of some event $A \in F$ is given by

$$J(A) = C \log (1/P(A)),$$

where

$$C = (\log 2)^{-1}.$$

This measure was first proposed by Shannon and Weaver [6]. A special case for events on the unit interval was given by Wiener [9].

To apply this measure of information in the above form, the probability measure P must be known explicitly. Since this is usually not true in practice, estimates of the Shannon-Wiener measure are obtained by replacing $P(A)$ with its relative frequency $n(A)/n$, where $n(A)$ denotes the number of times event A occurs in n independent trials. Since, by the Strong Law of Large Numbers

$$\frac{n(A)}{n} \to P(A) \qquad \text{a.s.}$$

$$\hat{J}_n(A) = C \log (n/n(A)) \to J(A) \qquad \text{a.s.}$$

The Shannon-Wiener measure can be adapted to the type of information feedback system currently under investigation by updating the relative frequency of event A after each cycle. In this application, the DM need not choose between competing COA's at the commencement of each new loop. He simply notes the occurrence or non-occurrence of event A and then decides whether or not to continue the feedback process for another cycle.

For fixed event $A \in F$ define

$$N_k = \text{number of times event } A \text{ occurs in the first } k \text{ cycles.}$$

Thus,

$$\hat{J}_k = C \log (k/N_k). \tag{15}$$

One problem with this estimate of the Shannon-Wiener information function is that $N_k = 0$ until event A occurs. Consequently, no estimate of $J(A)$ is possible until the first trial (call it k_1) on which event A occurs. Thus line (15) is valid only for $k \geqslant k_1$.

Since no information is directly measured on the first $k_1 - 1$ trials, no incremental information is possible. Thereafter,

$$\begin{aligned} X_{k_1} &= \hat{J}_{k_1} = C \log k_1 \\ X_k &= \hat{J}_k - \hat{J}_{k-1} \qquad k \geq k_1 + 1 \end{aligned} \tag{16}$$

Now if event A occurs on trial k, $N_k = N_{k-1} + 1$, and

$$X_k = C \log \left(\frac{k(N_{k-1})}{(k-1)N_k} \right). \tag{17}$$

Since $N_k \geq 1$ for $k \geq k_1$, the argument of the logarithm in (17) is less than or equal to 1, which implies that $X_k \leq 0$. If event A does not occur on the kth trial, $N_k = N_{k-1}$, and

$$X_k = C \log \left(\frac{k}{k-1} \right).$$

Clearly $X_k > 0$ in this instance. It follows, therefore, that

$$| X_k | = \begin{cases} C \log \left(\dfrac{(k-1)N_k}{k(N_{k-1})} \right) & \text{if } A \text{ occurs on trial } k \\ C \log \left(\dfrac{k}{k-1} \right) & \text{if } A^c \text{ occurs on trial } k, \end{cases} \tag{18}$$

and from (18) it is easy to show that $| X_k | < \delta$ implies

$$\begin{aligned} &\frac{(k-1)N_k}{k(N_{k-1})} < \lambda && \text{if event } A \text{ occurs on trial } k \\ &k/(k-1) < \lambda && \text{if event } A^c \text{ occurs on trial } k, \end{aligned} \tag{19}$$

where $\lambda = e^{\delta/c} = 2^{\delta}$.

The stopping rule given in Sec. 2 can be modified to apply in this situation as follows.

Stopping rule. Do not stop for any trial $k \leq k_1$. For $k > k_1$ stop the first time condition (19) is satisfied.

Stopping is not allowed on trial k_1 since $N_{k_1} = 1$ implies that the left-hand side of (19) is undefined.

Since the incremental information can be negative, we apply Theorem 2. As in the previous example, we satisfy condition (i) by establishing a bound for $E(T)$. To establish this bound, we first note that, if A^c occurs on trial k, stopping takes place if $k/(k-1) < \lambda$. But this implies that $k > \lambda/(\lambda - 1)$.

Let $M = [\lambda/(\lambda - 1)]$ denote the largest integer less than or equal to $\lambda/(\lambda - 1)$. Now, if stopping hasn't already occurred, it must take place the first time $k > \max(M, k_1)$ and event A^c occurs. If we denote this trial by T^*, then $T \leq T^*$ a.s.

Now define

$$L = T^* - \max(M, k_1).$$

L gives the location of T^* in terms of cycles beyond $\max(M, k_1)$. Now if $p = P(A)$ and

$q = P(A^c)$, random variable L has a geometric distribution whose mean is

$$E(L) = 1/q. \tag{20}$$

Random variable $\max(M, k_1)$ has a modified geometric distribution whose mean is

$$E[\max(M, k_1)] = M(1 - q^M) + \sum_{k=M+1}^{\infty} kpq^{k-1}.$$

After some manipulation, it can be shown that

$$E[\max(M, k_1)] = M + q^M/p. \tag{21}$$

From (20) and (21) it follows that

$$E(T^*) = M + q^M/p + 1/q.$$

Note also that $E(k_1) = 1/p$. Thus, since $k_1 \leqslant T < T^*$,

$$\frac{1}{p} < E(T) < M + q^M/p + 1/q. \tag{22}$$

The right-hand side of (22) is not necessarily a sharp bound for $E(T)$, since T may be considerably smaller than T^* if event A occurs with high probability.

To satisfy condition (ii) of Theorem 2 we first establish that

$$\frac{1}{2} < \frac{k(N_{k-1})}{(k-1)N_k} < 1 \qquad \text{a.s. for all } k > k_2$$

where k_2 denotes the trial on which event A occurs for the second time. From (17) it follows that $-1 < X_k < 0$ a.s. if event A occurs on trial k. In a similar manner, it can be established that $0 < X_k < 1$ a.s. if event A^c occurs on trial k. Thus condition (ii) holds with $\alpha = -1$ and $\beta = 1$. Condition (iii) is also immediate, since

$$\begin{aligned} p^+ &= P(X_k > 0) = q \\ p^- &= P(X_k \leqslant 0) = p \qquad \text{for all } k \leqslant T. \end{aligned}$$

Finally, we note that event $(T \geqslant k)$ depends on the outcomes of the first $k - 1$ trials, while event $(X_k > 0)$ is a function of the outcome of trial k. Since the trials are independent, condition (iv) is satisfied.

Since stopping cannot occur until after trial k_1,

$$\hat{J}_{T(A)} = \hat{J}_{k_2} + \sum_{j=1}^{T-k_2} X_{k_2+j} = \hat{J}_{k_2} + \hat{I}_{T-k_2}. \tag{23}$$

Therefore, applying Theorem 2 to random variable $T - k_2$ rather than T, we get

$$-pE(T - k_2) \leqslant E(\hat{I}_{T-k_2}) \leqslant q\, E(T - k_2). \tag{24}$$

From (15) and the fact that log x is concave downward we see that

$$0 \leqslant E(\hat{J}_{k_2}) < C \log E(k_2/2).$$

Since $E(k_2) = 2/p$ it follows from (23), (24), and the above inequality that

$$-pE(T) + 2 \leqslant E[\hat{J}_T(A)] \leqslant -C \log p + q\,E(T) - 2q/p.$$

Finally, from (22) and the fact that $J(A)$ cannot be negative, we obtain

$$0 \leqslant E(\hat{J}_T(A)) \leqslant Mq + 1 - q(2 - q^M)/p - C \log P.$$

Theorem 1 can also be applied to this example. Condition (ii) of Theorem 1 is satisfied for $\gamma = 1$. Thus, applying Theorem 1

$$E[\hat{I}_{T-k_2}] \leqslant E(T - k_2) = M + q^M/p + 1/q - 2/p,$$

and

$$0 \leqslant E[\hat{J}_T(A)] \leqslant M + 1/q - (2 - q^M)/p - C \log p.$$

Clearly, the upper bound obtained from Theorem 1 is greater than the one obtained from Theorem 2, since the coefficient of $E(T - k_2)$ is 1 with Theorem 1 and $q < 1$ with Theorem 2. This example illustrates why Theorem 2 is preferable when the increments can be negative.

4. CONDITIONS FOR THE EXPECTED STOPPING TIME TO BE FINITE

In the previous examples, we were able to show $E(T) < \infty$ by computing finite bounds for $E(T)$. Since this is not always possible, we now consider some necessary and sufficient conditions for the expected stopping time to be finite.

Obviously $E(T)$ cannot be finite unless stopping time T is finite with probability 1. Clearly, this condition holds whenever the incremental sequence $X_k \to 0$ a.s. since this convergence implies that

$$P(T = \infty) = P(\,|\,X_k\,| \geqslant \delta \text{ for all } k) = 0.$$

However, the a.s. convergence of X_k to zero is actually stronger than necessary. We will now proceed to show that it is necessary only that a sampling subsequence of the increments converge a.s. to zero.

Recall that probability measure P is defined over a measurable space (Ω, F), and denote by ω an element of Ω. Let

$$A_m = \left\{\omega\colon |\,X_{n_m(\omega)}(\omega)\,| < \frac{1}{m} \text{ for some } n_m(\omega)\right\},$$

where the indices are sampling random variables. That is, $1 \leqslant n_1 \leqslant n_2 \cdots \leqslant n_m \leqslant \cdots$ are integer-valued random variables defined on (Ω, F) such that $\{n_m = k\} \in F(X_1, \ldots, X_k)$. Now define

$$B = \{\omega\colon \text{there exists a sampling subsequence } X_{n_k} \text{ such that } \lim X_{n_k(\omega)}(\omega) = 0\}.$$

LEMMA 1. If $X_k \neq 0$ a.s. for all $k \leqslant T$, (25)

$$\text{then} \quad \bigcap_{m=1}^{\infty} A_m \subset B.$$

Proof. $\omega \in \bigcap_{m=1}^{\infty} A_m$ implies that for each m there exists

$$\text{some } n_m(\omega) \text{ such that } | X_{n_m(\omega)}(\omega) | < \frac{1}{m}. \tag{26}$$

Now let $S_\omega = \{n_m(\omega): m \geqslant 1\}$ where $n_m(\omega)$ satisfies condition (26) for fixed $\omega \in \Omega$.
Assume that S_ω is finite, and let

$$u = \min_{j_m \in S_\omega} | X_{j_m} |.$$

Condition (25) implies that either $u > 0$ or ω is contained in a set of P-measure zero. Therefore, excluding sets of measure zero, for all m such that $1/m < u$,

$$| X_{n_m(\omega)}(\omega) | \geqslant u > 1/m. \tag{27}$$

But $n_m \in S_\omega$ implies that $| X_{n_m} | < 1/m$ which contradicts (27). Thus S_ω must be infinite for all $\omega \in \Omega$.

Now, since S_ω is infinite, it is possible to pick an increasing subsequence.

$$1 < n_{k_1(\omega)} < n_{k_2(\omega)} < \cdots < n_{k_t(u)} \cdots$$

such that

$$| X_{n_{k_t(w)}}(w) | < \frac{1}{k_t}$$

Thus, $\omega \in \bigcap_{m=1}^{\infty} A_m$ implies that there exists a subsequence

$$\{X_{n_{k_t(\omega)}}; e > 1\} \quad \text{such that}$$

$$\lim_{t \to \infty} X_{n_{k_t(\omega)}}(\omega) = 0.$$

The conclusion of this lemma is then immediate from the definition of event B.

LEMMA 2: If condition (25) holds, $B \subset \bigcap_{m=1}^{\infty} A_m$.

Proof. $\omega \in B$ implies that for *any* integer m there exists $N_m(\omega)$ such that

$$| X_{n_k(\omega)}(\omega) | < \frac{1}{m} \text{ for all } n_k \geqslant N_m,$$

which implies that $\omega \in A_m$. Since m is chosen arbitrarily, it follows that $\omega \in B$ implies that $\omega \in A_m$ for all $m \geqslant 1$. The conclusion follows.

From Lemmas 1 and 2 it is clear that when condition (25) holds,

$$\bigcap_{m=1}^{\infty} A_m = B. \tag{28}$$

THEOREM 3. If condition (25) holds and if $E(T) < \infty$ for all $\delta > 0$, then there exists a sampling subsequence $\{X_{n_m}: m \geqslant 1\}$ such that

$$\lim_{m \to \infty} X_{n_m} = 0 \quad \text{a.s.}$$

Proof.

$$E(T) = \sum_{k=1}^{\infty} P(T \geqslant k) < \infty \quad \text{for all} \quad \delta$$

implies that $P(T \geqslant k \text{ i.o.}) = 0$ for all δ. This is an immediate consequence of the Borel-Cantelli lemma. Note that

$$\{T \geqslant k \text{ i.o.}\} = \{\omega: |X_n(\omega)| \geqslant \delta \quad \text{for all} \quad 1 \leqslant n \leqslant k-1 \quad \text{for infinitely many} \quad k\text{'s}.$$

Now suppose that for some integer n_0, $|X_{n_0}(\omega)| < \delta$. If this is true, $|X_n(w)| \geqslant \delta$ for all $1 < n < k-1$ only for $k \leqslant n_0$. It follows that

$$\{T \geqslant k \text{ i.o.}\} = \{\omega: |X_n(w)| \geqslant \delta \quad \text{for all} \quad n\} \quad \text{for all} \quad \delta > 0.$$

Thus setting $\delta = 1/m$,

$$\{T \geqslant k \text{ i.o.}\}^c = \{w: |X_{n_m}(w)| < 1/m \quad \text{some} \quad n_m\} = A_m.$$

But $P(T \geqslant k \text{ i.o.}) = 0$ for all $\delta \Rightarrow P(A_m) = 1$ for all m. Thus, from (28)

$$P(B) = P(\bigcap_{m=1}^{\infty} A_m) = 1, \tag{29}$$

and the Theorem is proved.

As a consequence of Theorem 3 it follows that, so long as condition (25) holds, a necessary condition for the expected stopping time to be finite for all thresholds $\delta > 0$ is that there exists a sampling subsequence of the increments that converges to zero a.s. Clearly, if $X_k \to 0$ a.s., all subsequences converge to zero a.s., but the convergence of the entire sequence of increments to zero is not necessary.

The argument used to prove Theorem 3 can be reversed to show that line (29) implies that $P(T > k \text{ i.o.}) = 0$. However, since the implication in the Borel-Cantelli lemma is one way, it cannot be proved in this manner that the existence of a sampling subsequence converging a.s. to zero is sufficient for $E(T)$ to be finite.

Any condition that guarantees the convergence of the series $\sum_{k=1}^{\infty} P(T \geqslant k)$ is sufficient for $E(T)$ to be finite. A rather interesting sufficient condition can be stated in terms of conditional probability.

THEOREM 4. If

$$\lim_{k \to \infty} P(T \geqslant k + 1 \mid T \geqslant k) = \lim_{k \to \infty} P(\,|X_k| \geqslant \delta \,\|\, X_n| \geqslant \delta \qquad 1 \leqslant n \leqslant k - 1)$$
$$= L < 1,$$

then

$$E(T) < \infty.$$

The proof of this theorem is an immediate consequence of the ratio test for convergent series and will be omitted. This theorem was not applicable to either example considered in this study because of the extreme difficulty of expressing $P(T \geqslant k + 1 \mid T \geqslant k)$ as a function of k. We suspect this difficulty will persist in other applications.

REFERENCES

1. R. Aló, R. Kleyle, and A. de Korvin, Decision making mechanisms and stopping times in a generalized information system. *Math. Modelling* **6**, 259–271 (1985).
2. B. K. Ghosh, *Sequential Tests of Statistical Hypotheses*. Addison-Wesley Publishing Co., Reading, MA (1970).
3. D. G. Hoel, M. Sobel, and G. H. Weiss, A survey of adaptive sampling for clinical trials. In *Perspectives in Biometry* (Edited by R. Elashoff), Academic Press, New York (1975).
4. R. Kleyle and A. de Korvin, Switching mechanisms in a generalized information system *Cybernetics and Systems: An International Journal* **15**, 145–167 (1984).
5. H. Robbins, Some aspects of the sequential design of experiments. *Bull. Amer. Math. Soc.* **58**, 527–535 (1952).
6. C. E. Shannon and W. Weaver, *A Mathematical Theory of Communication*, University of Illinois Press, Urbana, IL (1949).
7. M. Sobel, An essentially complete class of decision functions for certain standard sequential problems. *Annals Math. Stat.* **24**, 319–337 (1953).
8. A. Wald, *Sequential Analysis*, John Wiley and Sons, Inc., New York (1947).
9. N. Wiener, Cybernetics or control and communication in the animal and the machine. *Actualites Scientifiques et Industrielles*, No. 1053, Hermann et Cie, Paris (1948).
10. M. C. Yovits, C. R. Foulk, and L. L. Rose, Information flow and analysis: Theory, simulation and experiment. Part I. Basic theoretical and conceptual development. *J. Amer. Soc. Info. Sci.* **32**, 187–202 (1981).
11. M. C. Yovits, C. R. Foulk, and L. L. Rose, Information flow and analysis: Theory, simulation and experiment. Part II. Simulation, examples and results. *J. Amer. Info. Sci.* **32**, 203–210 (1981).
12. M. C. Yovits, C. R. Foulk, and L. L. Rose, Information flow and analysis: Theory, simulation and experiment. Part III. Preliminary experiments and analysis. *J. Amer. Soc. Info. Sci.* **32**, 243–248 (1981).